$1/N$-Expansion and the Dilute Bose Gas Beyond Mean-field Theory

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We consider the nonrelativistic interacting Bose gas at zero and finite temperature. Using the $1/N$-expansion, we derive expressions for the free energy density and the number density to next-to-leading order in $1/N$. Outside the critical region and at weak coupling, our calculations reduce to the well-known mean-field results for the dilute Bose gas. We also rederive the nonperturbative critical density for Bose condensation which was first calculated by Baym, Blaizot, Holzmann, Lalöe, and Vautherin.

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I. INTRODUCTION

The dilute Bose gas has a very long history dating back to the classic paper by Bogoliubov in 1947 [1]. At zero temperature, the quantum loop expansion is essentially an expansion in powers of the gas parameter $\sqrt{\rho a^3}$, where ρ is the number density and a is the s-wave scattering length. The leading quantum corrections to the mean-field results for the energy density and the speed of sound were calculated many years ago by Lee and Yang [2].

Finite-temperature corrections to the pressure of a dilute Bose gas were first calculated by Lee and Yang [3]. By performing an expansion of the pressure about zero temperature, they showed that the leading correction term goes as T^4. This reflects the fact that the thermodynamics at low temperature is completely dominated by the linear part of the Bogoliubov spectrum.

Most approaches to the thermodynamics of the dilute Bose gas are mean-field approximations [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Common to all mean-field calculations is that they predict a critical temperature T_c for Bose condensation which is the same as that of an ideal Bose gas [15]. Moreover some of the mean-field approximations such as the Popov appoximation predict a first-order phase transition. This is in disagreement with universality-class arguments which tell us that the NR Bose gas goes through a second-order phase transition since the it belongs to the universality class of the 3-dimensional $O(2)$-symmetric spin model. Another problem with some of the mean-field approximation such as the Hartree-Fock-Bogoliubov (HFB) approximation, is that the dispersion relation is gapped [5, 16]. This is in conflict with the Hugenholtz-Pines theorem [17, 18] which states that the spontaneous breaking of the $U(1)$ symmetry in the nonrelativistic (NR) Bose gas gives rise to a gapless mode. Note however, that this problem with the HFB approximation has recently been solved by Yukalov and Kleinert by introducing a separate chemical potential for the condensate [14] (see also Ref. [19] for a modfied gapless Hartree-Fock approach in the context of relativistic $O(N)$-symmetric theories).

Examples of calculations beyond mean-field theory are the renormalization group calculations of the spin-zero Bose gas of Refs. [20, 21, 22, 23]. Such calculations typically sum up classes of diagrams from all orders of perturbation theory and they show that the dilute Bose gas undergoes a second-order phase transitions in accordance with expectation based on universality [20, 21]. Other calcuations [24, 25] are based on the two-particle-irreducible (2PI) diagram effective-action approach first developed by Luttinger and Ward [26] and by Baym [27] in the context of nonrelativistic fermions. Combined with the $1/N$ expansion, the 2PI effective action is particular suited to deal with nonequilibrium phenomena [25], and has also been used successfully to calculate critical exponents [28]. Very recently, the equation of state of the dilute Bose gas was calculated by Pilati *et al* [29] using the Path Integral Monte Carlo method.

The problem of calculating the critical temperature for a dilute Bose gas has been around for half a century (See. Refs. [16, 30] for a review), butthe issue been was first settled 6-7 years ago by Baym *et al* [31, 32]. The Hartree-Fock (HF) approximation which amounts to perturbation theory to leading order in the scattering length merely implies a redefinition of the chemical potential and hence predicts no shift in the critical temperature. Going beyond the HF approximation using perturbation theory, one is facing infrared divergences which become more severe as one goes to higher orders. Thus perturbation theory breaks down close to T_c which shows that the physics of the phase transition is inherently nonperturbative. Using effective-field-theory methods, it was shown rigorously by Baym *et al* [31, 32] that the parametric dependence of the change in the critical temperature, ΔT_c, is linear in the scattering length a in the dilute limit:

$$\frac{\Delta T_c}{T_c^0} \;=\; c\left(\rho_c^0\right)^{1/3} a \;, \qquad (1)$$

where c is a constant, and T_c^0 and ρ_c^0 are the critical temperature and critical density of the ideal Bose gas, respectively. The coefficient c has been determined by analytical as well as numerical means in recent years. This includes effective-field theory [31, 32, 33], high-precision Monte-Carlo calculations [34, 35, 36, 37], simulations of classical field theory in the microcanonical ensemble [38], $1/N$-expansions [39, 40], variational perturbation theory [41], and the linear delta expansion [42, 43, 44].

The fact that mean-field theories and perturbation theory fail in the critical region, warrants a nonperturbative approach. In the present paper, we apply the $1/N$-expansion [45, 46] to the interacting NR Bose gas at finite temperature. In contrast with e.g. the HFB approximation, this approach satisfies the Hugenholtz-Pines theorem order by

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order in $1/N$. This expansion is a nonperturbative method that sums up infrared-divergent diagrams from all orders of perturbation theory. For example, a leading-order calculation sums up all daisy and superdaisy diagrams. Moreover, in order to get results beyond mean field, one must typically go to nex-to-leading order (NLO) in the $1/N$ expansion. Following the approach in Ref. [47], we derive the thermodynamic quantities at finite temperature to next-to-leading order in the $1/N$-expansion.

The paper is organized as follows. In Sec. II, we briefly discuss the nonrelativistic Bose gas and the $1/N$-expansion. In Sec. III, we discuss the thermodynamic potential and the gap equations. We also derive the leading-order and next-to-leading-order results. We summarize in Sec. IV.

II. DILUTE BOSE GAS AND EFFECTIVE ACTION

The Euclidean Lagrangian for a nonrelativistic Bose gas with N species is

$$\mathcal{L} = \hbar\psi_i^\dagger \partial_\tau \psi_i + \frac{\hbar^2}{2M}\nabla\psi_i^\dagger \cdot \nabla\psi_i - \mu\psi_i^\dagger\psi_i + \frac{g}{2N}(\psi_i^\dagger\psi_i)^2 , \tag{2}$$

where $i = 1, 2, ..., N$, ψ_i is a complex field, μ is the chemical potential[1] and M is the mass of the bosons. In the following we will be using units where $2M = \hbar = 1$ and summation over repeated indices is understood. The Lagrangian (2) is invariant under the group $O(2N)$. For $N = 1$, this reduces to the Lagrangian for the nonrelativistic Bose gas, and we can identify the coupling constant with the s-wave scattering length, $g = 8\pi a$ [2]. In order to eliminate the quartic interaction from the Lagrangian we introduce an auxiliary field α and add to Eq. (2) the following term

$$\mathcal{L}_\alpha = \frac{2N}{g}\left[\alpha - \frac{ig}{2N}\left(\psi_i^\dagger\psi_i - \frac{N\mu}{g}\right)\right]^2 . \tag{3}$$

The Lagrangian can now be written as

$$\mathcal{L} = \frac{2N}{g}\alpha^2 - 2i\alpha\left(\psi_i^\dagger\psi_i - \frac{N\mu}{g}\right) + \psi_i^\dagger\partial_\tau\psi_i + \nabla\psi_i^\dagger \cdot \nabla\psi_i - \frac{N\mu^2}{2g} . \tag{4}$$

By using the equation of motion for α, one can eliminate this field altogether and one recovers the original Lagrangian (2).

Integrating over the fields $\psi_2, \psi_3...\psi_N$, we obtain the effective action for α and ψ_1:

$$S_{\text{eff}} = (N-1)\text{Tr}\ln\left[\partial_\tau - \nabla^2 - 2i\alpha\right]$$
$$+ \int_0^\beta d\tau \int d^3x \left[\psi_1^\dagger\partial_\tau\psi_1 + \nabla\psi_1^\dagger \cdot \nabla\psi_1 + \frac{2N}{g}\alpha^2\right.$$
$$\left. -2i\alpha\left(\psi_1^\dagger\psi_1 - \frac{N\mu}{g}\right) - \frac{N\mu^2}{2g}\right] , \tag{5}$$

where $\beta = 1/T$. In Eq. (5), Tr implies taking the trace of the differential operator inside the brackets. We next parametrize the fields ψ_1 and α by writing them as sums of space-time independent expectation values ϕ_0 and im and quantum fluctuating fields $\tilde\psi$ and $\tilde\alpha$:

$$\psi_1 = \sqrt{N}\phi_0 + \tilde\psi_1 , \tag{6}$$
$$\alpha = im + \frac{\tilde\alpha}{\sqrt{N}} . \tag{7}$$

The rescaling with factors of $1/\sqrt{N}$ is just facilitate the counting of factors of N. The field ϕ_0 can be rotated so it is real. It can be shown that the expectation value of α is purely imaginary so m is real. Substituting Eqs. (6)–(7) into Eq. (5), we obtain

$$S_{\text{eff}} = (N-1)\text{Tr}\ln\left[\partial_\tau - \nabla^2 + 2m - \frac{2i\tilde\alpha}{\sqrt{N}}\right]$$
$$+ \int_0^\beta d\tau \int d^3x \left[\tilde\psi_1^\dagger\partial_\tau\tilde\psi_1 - \nabla\tilde\psi_1^\dagger \cdot \nabla\tilde\psi_1\right.$$
$$+ \frac{2N}{g}\left(im + \frac{\tilde\alpha}{\sqrt{N}}\right)^2 + 2\left(m - \frac{i\tilde\alpha}{\sqrt{N}}\right)$$
$$\times \left(N\phi_0^2 + \sqrt{N}\phi_0(\tilde\psi_1^\dagger + \tilde\psi_1) + \tilde\psi_1^\dagger\tilde\psi_1\right.$$
$$\left.\left. - \frac{N\mu}{g}\right) - \frac{N\mu^2}{2g}\right] , \tag{8}$$

where we have set $k_B = 1$. Writing $\tilde\psi_1 = (\phi_1 + i\phi_2)/\sqrt{2}$ and expanding Eq. (8) in powers of $1/\sqrt{N}$ through order $1/N$, we obtain

$$\frac{S_{\text{eff}}}{\beta V} = (N-1)\sumint_P \ln\left[ip_0 + p^2 + 2m\right] + 2mN\phi_0^2$$
$$- N\frac{(\mu + 2m)^2}{2g}$$
$$+ \frac{1}{2}\sumint_P \chi^T(-P)\begin{pmatrix} p^2 + 2m & p_0 & -2\sqrt{2}i\phi_0 \\ -p_0 & p^2 + 2m & 0 \\ -2\sqrt{2}i\phi_0 & 0 & \frac{4}{g} + 4\Pi(P,m) \end{pmatrix}\chi(P) , \tag{9}$$

where V is the volume of the system, $\chi = (\phi_1(P), \phi_2(P), \tilde\alpha(P))$ are the Fourier transforms of ϕ_1, ϕ_2, and $\tilde\alpha$. We have neglected linear terms that vanish at the minimum of the thermodynamic potential. We have here introduced the sum-integral

$$\sumint_Q \equiv T\sum_{q_0 = 2\pi nT}\int\frac{d^3q}{(2\pi)^3} . \tag{10}$$

The integral over three-momentum is regularized by using a three-dimensional cutoff Λ. Finally, the function $\Pi(P,m)$ is defined by

$$\Pi(P,m) = \sumint_Q \frac{1}{q^2 + 2m + iq_0}\frac{1}{(p+q)^2 + 2m - i(p_0+q_0)} . \tag{11}$$

III. THERMODYNAMIC POTENTIAL AND GAP EQUATIONS

The thermodynamic potential Ω is given by all one-particle irreducible diagrams and is given by a series in $1/N$. The

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[1] In principle, one could introduce N different chemical potentials μ_i, but this is not natural in this context. The symmetry group would then be $[O(2)]^N$.

[2] Note that the expansion parameter formally is $1/2N$ so the expansion makes sense also for $N = 1$.

first two terms are easily obtained from Eq. (9) by gaussian integration over χ. We can then write

$$\Omega = N\Omega_{\text{LO}} + \Omega_{\text{NLO}} , \tag{12}$$

where

$$\Omega_{\text{LO}} = -\frac{(\mu + 2m)^2}{2g} + 2m\phi_0^2$$
$$+ \sum\!\!\!\!\!\!\int_P \ln\left[ip_0 + p^2 + 2m\right] , \tag{13}$$
$$\Omega_{\text{NLO}} = \frac{1}{2}\sum\!\!\!\!\!\!\int_P \ln\left[\Pi(p,m) + \frac{1}{g} + \frac{2\phi_0^2(p^2 + 2m)}{p_0^2 + (p^2 + 2m)^2}\right] . \tag{14}$$

The free energy $\mathcal{F}$ is given by all connected diagrams and is independent of the condensate ϕ_0 and the expectation value m. So if the thermodynamic potential is evaluated at the values of ϕ_0 and m, which satisfy the equations

$$\frac{\partial\Omega}{\partial\phi_0} = 0 , \tag{15}$$
$$\frac{\partial\Omega}{\partial m} = 0 , \tag{16}$$

the one-particle reducible diagrams vanish. The values of ϕ_0 and m, which satisfy the stationarity conditions (15) and (16) can be written as series in $1/N$:

$$\phi_0 = \phi_{0,\text{LO}} + \frac{1}{N}\phi_{0,\text{NLO}} , \tag{17}$$
$$m = m_{\text{LO}} + \frac{1}{N}m_{\text{NLO}} . \tag{18}$$

By series expanding the thermodynamic potential around the LO solution to the gap equations, the free energy can then be written as

$$\mathcal{F} = N\Omega_{\text{LO}}(m_{\text{LO}}, \phi_{0,\text{LO}}) + \Omega_{\text{NLO}}(m_{\text{LO}}, \phi_{0,\text{LO}})$$
$$+ m_{\text{NLO}}\frac{\partial\Omega_{\text{LO}}}{\partial m}\bigg|_{m=m_{\text{LO}}, \phi_0 = \phi_{0,\text{LO}}}$$
$$+ \phi_{0,\text{NLO}}\frac{\partial\Omega_{\text{LO}}}{\partial\phi_0}\bigg|_{m=m_{\text{LO}}, \phi_0 = \phi_{0,\text{LO}}} + \mathcal{O}\left(1/N^2\right) . \tag{19}$$

The free energy to NLO then reduces to

$$\mathcal{F} = N\mathcal{F}_{\text{LO}} + \mathcal{F}_{\text{NLO}} , \tag{20}$$

where

$$\mathcal{F}_{\text{LO}} = \Omega_{\text{LO}}(m_{\text{LO}}, \phi_{0,\text{LO}}) , \tag{21}$$
$$\mathcal{F}_{\text{NLO}} = \Omega_{\text{NLO}}(m_{\text{LO}}, \phi_{0,\text{LO}}) . \tag{22}$$

The number density ρ is given by the expectation value

$$\rho = \langle\psi^*\psi\rangle . \tag{23}$$

From the path-integral representation of the free energy $\mathcal{F}$

$$e^{-\beta V\mathcal{F}} = \int \mathcal{D}\psi^*\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3x \mathcal{L}} , \tag{24}$$

and Eq. (2), the expression for the number density can be written as

$$\rho = -\frac{\partial\mathcal{F}}{\partial\mu} . \tag{25}$$

Using Eqs. (4) and (25), one finds the following exact result

$$\rho = N\frac{\mu + 2m}{g} . \tag{26}$$

A. Leading-order results

At leading order the gap equations (15) and (16) become

$$4m\phi_0 = 0 , \tag{27}$$
$$\phi_0^2 = \frac{\mu + 2m}{g} - \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2 + 2m} . \tag{28}$$

The solution to Eq. (27) is that either m or ϕ_0 vanishes. $m = 0$ corresponds to the Bose-condensed phase and we discuss this first.

Substituting $m = 0$ into Eq. (13), the leading-order contribution to the free energy reduces to

$$\mathcal{F}_{\text{LO}} = -\frac{\mu^2}{2g} + \sum\!\!\!\!\!\!\int_P \ln\left[ip_0 + p^2\right] . $$

The sum-integral can be calculated by first writing the sum over Mastubara frequencies as a contour integral. By performing this contour integral and integrating over angles, we are left with

$$\mathcal{F}_{\text{LO}} = -\frac{\mu^2}{2g} + \frac{1}{4\pi^2}\int_0^\Lambda dp\, p^2\left[p^2 + 2T\ln\left(1 - e^{-\beta p^2}\right)\right]$$
$$= -\frac{\mu^2}{2g} + \frac{\Lambda^5}{20\pi^2} - 4\pi\left(\frac{T}{4\pi}\right)^{5/2}\zeta\left(\tfrac{5}{2}\right) . \tag{29}$$

The divergent term is eliminated by adding a vacuum counterterm $\Delta\mathcal{F}$ to the free energy $\mathcal{F}$ and the renormalized LO free energy is

$$\mathcal{F}_{\text{LO}} = -\frac{\mu^2}{2g} - 4\pi\left(\frac{T}{4\pi}\right)^{5/2}\zeta\left(\tfrac{5}{2}\right) . \tag{30}$$

Note that the LO free energy density does not agree with the standard mean-field result. The first term in Eq. (30) is the zero-temperature result in the mean-field approximation, while the second term is the finite-temperature contribution for an ideal Bose gas. The LO number density in terms of the condensate ϕ_0 follows from Eqs. (26) and (28)

$$\rho = \phi_0^2 + \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2} . \tag{31}$$

The sum-integral in the gap equation (31) can be calculated in a similar manner as the one appearing in the free energy and one obtains

$$\rho = \phi_0^2 + \frac{\Lambda^3}{12\pi^2} - \left(\frac{T}{4\pi}\right)^{3/2}\zeta\left(\tfrac{3}{2}\right) . \tag{32}$$

The divergent term reflects the fact that the expectation value of the number density operator also has divergences and we need to add a counterterm $\delta\rho$ to eliminate them [48]. This yields

$$\rho = \phi_0^2 + \left(\frac{T}{4\pi}\right)^{3/2}\zeta\left(\tfrac{3}{2}\right) . \tag{33}$$

The critical temperature is found by setting $\phi_0 = 0$. This yields

$$T_c^0 = 4\pi\left[\frac{n}{\zeta\left(\tfrac{3}{2}\right)}\right]^{2/3} , \tag{34}$$

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which is the standard result for an ideal Bose gas [15]. Inverting Eq. (45) then gives the critical number density ρ_c^0 for Bose condensation

$$\rho_c^0 \;=\; \zeta\left(\tfrac{3}{2}\right)\left[\frac{T}{4\pi}\right]^{3/2} . \tag{35}$$

We next discuss the symmetric phase where $\phi_0 = 0$. The free energy in the symmetric phase is given by

$$\begin{aligned}
\mathcal{F} \;&=\; \sum\!\!\!\!\!\!\int_P \ln\left[ip_0 + p^2 + 2m\right] - \frac{1}{2}g\left(\sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2 + 2m}\right)^2 \\
&=\; -4\pi\left(\frac{T}{4\pi}\right)^{5/2}\mathrm{Li}_{5/2}\left(e^{-2\beta m}\right) \\
&\quad -\frac{1}{2}g\left(\frac{T}{4\pi}\right)^3\mathrm{Li}_{3/2}^2\left(e^{-2\beta m}\right) ,
\end{aligned} \tag{36}$$

where $\mathrm{Li}_n(x) = \sum_k x^k/k^n$ is the polylogarithmic function and we have have eliminated $\mu + 2m$ in Eq. (13) using Eq. (16). In the symmetric phase, the number density reduces to

$$\begin{aligned}
\rho \;&=\; \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2 + 2m} \\
&=\; \left(\frac{T}{4\pi}\right)^{3/2}\mathrm{Li}_{3/2}\left(e^{-2\beta m}\right) ,
\end{aligned} \tag{37}$$

where we have neglected a temperature-independent divergent term. The number density in the symmetric phase can be interpreted as that of an ideal Bose gas with an effective chemical potential $\mu_{\mathrm{eff}} = -2m$. Eq. (28) can be interpreted as a self-consistent one-loop equation for the chemical potential and the critical number density for Bose condensation is then given by $\mu_{\mathrm{eff}} = 0$.

B. Next-to-leading order results

The NLO gap equations (15)–(16) are

$$0 \;=\; \left[m + \frac{1}{2N}\sum\!\!\!\!\!\!\int_P \frac{1}{\Pi(P,0) + \frac{1}{g} + \frac{2\phi_0^2 p^2}{p_0^2 + p^4}}\frac{p^2}{p_0^2 + p^4}\right]\phi_0 , \tag{38}$$

$$\begin{aligned}
\phi_0^2 \;&=\; \frac{\mu + 2m}{g} - \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2 + 2m} \\
&\quad -\frac{1}{N}\sum\!\!\!\!\!\!\int_P \frac{1}{\Pi(P,0) + \frac{1}{g} + \frac{2\phi_0^2 p^2}{p_0^2 + p^4}} \\
&\quad \times\left[\frac{1}{4}\frac{d\Pi(P,0)}{dm} + \frac{\phi_0^2}{p_0^2 + p^4} - \frac{2p^4\phi_0^2}{(p_0^2 + p^4)^2}\right] ,
\end{aligned} \tag{39}$$

where we have used that it is consistent to set $m = 0$ in the NLO terms. Setting $N = 1$, the expression for the free energy now becomes

$$\begin{aligned}
\mathcal{F} \;&=\; -\frac{\mu^2}{2g} + \sum\!\!\!\!\!\!\int_P \ln\left[ip_0 + p^2\right] \\
&\quad +\frac{1}{2}\sum\!\!\!\!\!\!\int_P \ln\left[\Pi(p,0) + \frac{1}{g} + \frac{2\phi_0^2 p^2}{p_0^2 + p^4}\right] ,
\end{aligned} \tag{40}$$

where ϕ_0 satisfies the LO gap equation (28) and is to be considered a function of μ. The number density can be derived

from (40) and reads

$$\rho \;=\; \frac{\mu}{g} + \frac{1}{g}\sum\!\!\!\!\!\!\int_P \frac{1}{\Pi(P,0) + \frac{1}{g} + \frac{2\phi_0^2 p^2}{p_0^2 + p^4}}\frac{p^2}{p_0^2 + p^4} \tag{41}$$

Another more useful expression for the number density follows from Eqs. (26) and (39)

$$\begin{aligned}
\rho \;&=\; \phi_0^2 + \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2 + 2m} \\
&\quad +\sum\!\!\!\!\!\!\int_P \frac{1}{\Pi(P,0) + \frac{1}{g} + \frac{2\phi_0^2 p^2}{p_0^2 + p^4}} \\
&\quad \times\left[\frac{1}{4}\frac{d\Pi(P,0)}{dm} + \frac{\phi_0^2}{p_0^2 + p^4} - \frac{2p^4\phi_0^2}{(p_0^2 + p^4)^2}\right] . \tag{42}
\end{aligned}$$

The $1/N$ expansion is not limited to a weakly interacting Bose gas. If we make the additional assumption of weak coupling, we can make further approximations by simply neglecting $g\Pi(P,m)$. This approximation is valid outside the critical region and the free energy reduces to

$$\mathcal{F} \;=\; -\frac{\mu^2}{2g} + \frac{1}{2}\sum\!\!\!\!\!\!\int_P \ln\left[p_0^2 + \epsilon^2(p)\right] , \tag{43}$$

where

$$\epsilon(p) \;=\; p\sqrt{p^2 + 2g\phi_0^2} , \tag{44}$$

is the well-known Bogoliubov spectrum [1]. Similarly, the number density reduces to

$$\rho \;=\; \phi_0^2 + \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2} + g\phi_0^2\sum\!\!\!\!\!\!\int_P \frac{1}{p_0^2 + \epsilon^2(p)}\frac{p_0^2 - p^4}{p_0^2 + p^4} . \tag{45}$$

After performing the sum over Matsubara frequencies, we obtain

$$\rho \;=\; \phi_0^2 + \frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\frac{\epsilon^2(p)/p^2 + p^2}{\epsilon(p)}\left[1 + 2n(\epsilon(p))\right] , \tag{46}$$

where $n(x) = 1/(e^{\beta x} - 1)$ is the Bose-Einstein distribution function. At $T = 0$, this reduces to the Bogoliubov approximation and gives the standard weak-coupling result for the depletion of the condensate [2]:

$$\rho \;=\; \phi_0^2\left[1 + \frac{8}{3}\sqrt{\frac{\phi_0^2 a^3}{\pi}}\right] . \tag{47}$$

At finite temperature, this approximation goes beyond the Bogoliubov approximation since we via Eq. (45) take into account the temperature dependence of the condensate ϕ_0 when evaluating (43) [16].

In order to find the critical number density ρ_c, we must evaluate the right-hand side of Eq. (46) in the limit $\phi_0 \to 0$. The Bogoliubov spectrum then reduces to the free-particle spectrum and we recover the mean-field result (35).

We have seen that by neglecting $\Pi(P,m)$, one recovers the mean-field results for the dilute Bose gas. In order to go beyond mean field, we must keep the self-energy in the gap equations. Since the phase transition is dominated by the nonperturbative long-distance physics, the dominant contribution comes from the static Matsubara mode [31, 32, 40]. The

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contributions from the nonstatic Matsubara modes can be treated perturbatively and this fact underlies the calculations of Refs. [31, 32, 40] using dimensional-reduction techniques. If we denote the contribution to the self-energy $\Pi(P,m)$ from the zeroth Matsubara frequency at $p_0 = 0$ by $\Pi_0(p,m)$, we have

$$\Pi_0(p,0) = T \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} \frac{1}{(\mathbf{p}+\mathbf{q})^2} , \qquad (48)$$

$$\left. \frac{d\Pi_0(p,m)}{dm} \right|_{m=0} = -8T \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^4} \frac{1}{(\mathbf{p}+\mathbf{q})^2} . \qquad (49)$$

The critical number density is then found by settting $\phi_0 = 0$ and inserting (49) into (42) and rearranging terms. This yields

$$\rho_c = \sum\!\!\!\!\!\!\int_P \frac{1}{ip_0 + p^2} - 2T^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^4}$$
$$\times \int \frac{d^3p}{(2\pi)^3} \frac{1}{\Pi_0(p,m)+1/g} \frac{1}{(\mathbf{p}+\mathbf{q})^2} . \qquad (50)$$

The term

$$\Sigma(q) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\Pi_0(p,m)+1/g} \frac{1}{(\mathbf{p}+\mathbf{q})^2} \qquad (51)$$

can be interpreted as the leading correction to the self-energy function in the $1/N$ expansion. The corresponding string of Feynman diagrams is shown in Fig. 1. The first of these dia-
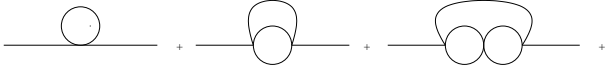

FIG. 1: Diagrams contributing to the self-energy at LO in the $1/N$ expansion.

grams corresponds to the HF approximation. The self energy needs mass renormalization, which is carried out by subtracting $\Sigma(0)$ [49]. Similarly, the second term in Eq. (50) can be interpreted in terms of Feynman diagrams [39, 40]. These are shown in Fig. 2.
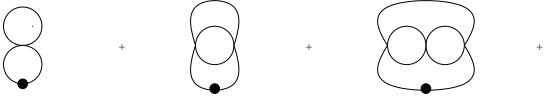

FIG. 2: Diagrams contributing to the number density at NLO in the $1/N$ expansion.

The first term in (50) is ρ_c^0, while the the second term was first calculated by Baym, Blaizot, and Zinn-Justin [39], and later by Arnold and Tomášik [40]. It is given by $gT/32\pi^2$ and the critical number density reduces to

$$\rho_c = \rho_c^0 - \frac{1}{2}g\left(\frac{T}{4\pi}\right)^2$$
$$= \rho_c^0 \left[1 - \frac{4\pi \left(\rho_c^0\right)^{1/3} g}{\left[\zeta(\frac{3}{2})\right]^{4/3}} \right] , \qquad (52)$$

where we have used $g = 8\pi a$. Thus the critical number density decreases linearly with the scattering length a. Instead of calculating the critical number density at fixed temperature, we can calculate the critical temperature at fixed number density by using

$$\frac{\Delta T_c}{T_c^0} \simeq -\frac{2}{3}\frac{\rho_c - \rho_c^0}{\rho_c^0} . , \qquad (53)$$

which is valid to leading order in the number-density expansion [32, 40]. This yields

$$\frac{\Delta T_c}{T_c^0} = \frac{8\pi}{3\left[\zeta(\frac{3}{2})\right]^{4/3}} \left(\rho_c^0\right)^{1/3} a$$
$$\approx 2.33 \left(\rho_c^0\right)^{1/3} a . \qquad (54)$$

It is interesting to note [39] that the results (52) and (54) are independent of N, although they are correct only in the limit $N \to \infty$. The value 2.33 of the constant c compares reasonably well with $c \approx 1.3$ from lattice calculations [34, 35, 36, 37] and classical field-theory simulations [38].

IV. SUMMARY

In this paper, we have used the $1/N$-expansion to derive thermodynamic quantities of the interacting Bose gas. If we make the additional assumption of weak coupling, we obtain the standard results for the dilute Bose gas. If we neglect the contribution to the number density from the nonstatic Matsubara modes at criticality, we obtain the result for the critical number density first obtained by Baym *et al* [31, 32] The $1/N$ expansion gives a coherent description of the dilute Bose gas in the critical region as well as outside it, where mean-field theories are normally sufficient.

We have not calculated the critical exponents associated with the second-order phase transition of the Bose gas. However, making the same approximations as we did for our calculations of the critical number density, one expects to obtain the standard results for the 1PI effective action to NLO in $1/N$ [49].

It is natural to ask whether the $1/N$ expansion is reliable for $N = 1$. In the case of the critical temperature for Bose condensation, the $1/N$ correction was calculated by Arnold and Tomasik [40] and the correction was approximately 26%. This is as expected: a correction of $1/2N$ or 50% multiplied by a constant of order one.

Acknowledgment

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